Section 8.1 Modeling with Differential Equations

General Differential Equations

A differential equation is an equation that contains an unknown function and one or more of its derivatives. The order of a differential equation is the order of the highest derivative that occurs in the equation.

A function f is called a solution of a differential equation if the equation is satisfied when $y = f(x)$ and its derivatives are substituted into the equation. When asked to solve a differential equation, we are expected to find all possible solutions of the equation.

When given an initial condition ($y(t_0) = y_0$), the problem of finding a solution of the differential equation that satisfies the initial condition is called an initial-value problem (IVP).

Models of Population Growth

One model for the growth of a population is based on the assumption that the population grows at a rate proportional to the size of the population: $\frac{dP}{dt} = kP$, where t represents the independent variable time, P represents the population, and k is some constant. If the population is nonzero and $k > 0$, then $P'(t) > 0$ for all t, which means the population is always increasing. This is unrealistic since many populations start by increasing in an exponential manner, but the population levels off when it approaches its carrying capacity K. A better model is given by the logistic equation:

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{K}\right).$$

Notice here that if $P = 0$ or $P = K$, $\frac{dP}{dt} = 0$. These are called equilibrium solutions.

Model for the Motion of a Spring

By Hooke's Law,

$$\text{restoring force } = -kx.$$

Ignoring external resisting forces (friction, etc.), by Newton's Second Law (force = mass $\times$ acceleration), we have

$$m\frac{d^2x}{dt^2} = -kx \text{ or, equivalently } mx'' + kx = 0.$$

Newton's Law of Cooling

Newton's Law of Cooling states that the rate of cooling of an object is proportional to the temperature difference between the object and its surroundings, provided that this difference is not too large. Let $T(t)$ be the temperature of the object at time t and T_S be the temperature of the surroundings, then Newton's Law of Cooling states:

$$\frac{dT}{dt} = k(T - T_S).$$