Power Series

Power Series A power series is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

where x is a variable and the c_ns are constants (depending on n) called the coefficients of the series.

More generally, a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

is called a power series centered at a.

(For each fixed x, the power series is a series of numbers which we can test for convergence.)

Theorem For a given power series $\sum_{n=0}^{\infty} c_n (x-a)^n$, there are only three possibilities:

(i) The series converges only when $x = a$.

(ii) The series converges for all x.

(iii) There is a positive number R such that the series converges if $|x-a| < R$ and diverges if $|x-a| \geq R$.

The number R in Case (iii) is called the radius of convergence of the power series. By convention, the radius of convergence is $R = 0$ in Case (i) and $R = \infty$ in Case (ii). The interval of convergence of a power series is the interval that consists of all values of x for which the series converges. In Case (i) the interval consists of just a single point a. In Case (ii) the interval is $(-\infty, \infty)$. In Case (iii) note that the inequality $|x-a| < R$ can be rewritten as $a - R < x < a + R$. When x is an endpoint of the interval, that is, $x = a \pm R$, anything can happen - the series might converge at one or both endpoints or it might diverge at both endpoints. Thus, in Case (iii) there are four possibilities for the interval of convergence:

$$(a-R, a+R), \quad (a-R, a+R], \quad [a-R, a+R), \quad [a-R, a+R]$$

The Ratio Test can be used to determine the radius of convergence R in most cases. The Ratio Test always fails when x is an endpoint of the interval of convergence, so the endpoints must be checked with some other test.