Section 9.3 Infinite Series

If we try to add the terms of an infinite sequence $\{a_n\}_{n=1}^{\infty}$ we get an expression of the form

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

which is called an infinite series and is denoted by

$$\sum_{n=1}^{\infty} a_n.$$

To determine whether or not a general series has a sum, we consider the partial sums:

$$\begin{aligned} s_1 &= a_1 \\ s_2 &= a_1 + a_2 \\ s_3 &= a_1 + a_2 + a_3 \\ &\vdots \\ s_n &= a_1 + a_2 + \cdots + a_n = \sum_{i=1}^{n} a_i \end{aligned}$$

These partial sums form a new sequence $\{s_n\}$. If $\lim_{n\to\infty} s_n = s$ (ie, if $\{s_n\}$ is a convergent sequence), then $\sum_{n=1}^{\infty} a_n = s$ and we call $\sum_{n=1}^{\infty} a_n$ a convergent series. If the sequence $\{s_n\}$ is divergent, then the series is also divergent.

Geometric Series

$$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1} \quad (a \neq 0)$$

$r = 1$

$$s_n = a + a + a + \cdots + a = na \to \pm\infty$$

$r \neq 1$

$$\begin{aligned} s_n &= a + ar + ar^2 + \cdots + ar^{n-1} \implies \\ rs_n &= ar + ar^2 + ar^3 + \cdots + ar^n \implies \\ s_n - rs_n &= a - ar^n \implies \\ s_n &= \frac{a(1-r^n)}{1-r} \end{aligned}$$

If $-1 < r < 1$, $r^n \to 0$ as $n \to \infty$, so

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{a(1-r^n)}{1-r} = \frac{a}{1-r}.$$

Also, if $|r| > 1$, $|r|^n \to \infty$ as $n \to \infty$ so $\lim_{n\to\infty} s_n$ diverges. The geometric series

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots$$

is convergent if $|r| < 1$ and its sum is

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}, \quad |r| < 1.$$

If $|r| \geq 1$, the geometric series is divergent.

Theorem The harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

is divergent.