Representations of Functions as Power Series

Theorem If the power series $\sum c_n(x-a)^n$ has radius of convergence $R > 0$, then the function f defined by

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots = \sum_{n=0}^{\infty} c_n(x-a)^n$$

is differentiable (and therefore continuous) on the interval $(a-R, a+R)$ and

(i) $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \displaystyle\sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$

(ii) $\displaystyle\int f(x)dx = C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \cdots = C + \sum_{n=0}^{\infty} c_n\frac{(x-a)^{n+1}}{n+1}$

Note 1 Equations (i) and (ii) can be rewritten in the form

(iii) $\displaystyle\frac{d}{dx}\left[\sum_{n=0}^{\infty} c_n(x-a)^n\right] = \sum_{n=0}^{\infty}\frac{d}{dx}[c_n(x-a)^n]$

(iv) $\displaystyle\int\left[\sum_{n=0}^{\infty} c_n(x-a)^n\right]dx = \sum_{n=0}^{\infty}\int c_n(x-a)^n dx.$

Note 2 Although the radius of convergence remains the same when a power series is differentiated or integrated, this does not mean that the interval of convergence remains the same. You must check the endpoints.