Section 7.5 Integrating Rational Functions By Partial Fraction Decomposition

How to Integrate Rational Functions:

Let $f(x) = \dfrac{P(x)}{Q(x)}$ be a rational function (ie, a ratio of polynomials).

- If $deg(P) \geq deg(Q)$, use polynomial long division to rewrite $f(x)$ as the sum of simpler fractions:

$$f(x) = \frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)} \text{ (R(x) remainder)}$$

- If $deg(P) < deg(Q)$, use partial fraction decomposition to rewrite $f(x)$ as the sum of simpler fractions.

Method of Partial Fraction Decomposition

If $deg(P) < deg(Q)$, we need to express $f(x)$ as a sum of partial fractions. We have 4 cases depending on $Q(x)$:

Case I The denominator $Q(x)$ is a product of distinct linear factors. This means we can write

$$Q(x) = (a_1x + b_1) \cdots (a_kx + b_k)$$

where no factor is repeated. In this case the partial fraction theorem states that there exists constants $A_1 \cdots A_k$ such that

$$\frac{P(x)}{Q(x)} = \frac{A_1}{a_1x + b_1} + \cdots + \frac{A_k}{a_kx + b_k}.$$

Case II $Q(x)$ is a product of linear factors, some of which are repeated. Suppose the first linear factor (a_1x+b_1) is repeated r times. Then instead of the single term $\dfrac{A_1}{a_1x + b_1}$, we would use

$$\frac{A_1}{(a_1x + b_1)} + \frac{A_2}{(a_1x + b_1)^2} + \cdots + \frac{A_r}{(a_1x + b_1)^r}.$$

Case III $Q(x)$ contains irreducible quadratic factors, none of which is repeated. If $Q(x)$ has the factor ax^2+bx+c, where $b^2 - 4ac < 0$, the expression for $\dfrac{P(x)}{Q(x)}$ will have a term of the form $\dfrac{Ax + B}{ax^2 + bx + c}$.

Case IV $Q(x)$ contains a repeated irreducible quadratic factor. Suppose the irreducible quadratic factor $ax^2 + bx + c$ is repeated r times. Then instead of the single term $\dfrac{Ax + B}{ax^2 + bx + c}$, we would use

$$\frac{A_1x + B_1}{(ax^2 + bx + c)} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_rx + B_r}{(ax^2 + bx + c)^r}.$$