Section 6.2 & 6.3 Volume

Cross Sections Volume can be computed by multiplying the common cross-sectional area and the height. When the cross-sectional areas are not constant, we need to use calculus.

Volume of a Solid with Known Cross-Sectional Area A solid S with cross-sectional area $A(x)$ at each point perpendicular to the x-axis on the interval $[a, b]$ has volume

$$V = \int_a^b A(x)\, dx.$$

Volumes of Revolution: Disks and Washers A solid of revolution is a solid figure S obtained by revolving a region R in the xy-plane about a line L (called the axis of revolution) that lies outside R. Note that such a solid S may be thought of as having circular cross sections in the direction perpendicular to L.

The Disk Method The disk method is used to find a volume generated when a region R is revolved about an axis L that is perpendicular to a typical approximating strip in R. Suppose R is the region bounded by the curve $y = f(x)$, the x-axis, and the vertical lines $x = a$ and $x = b$. Then if R is revolved about the x-axis, it generates a solid with volume

$$V = \int_a^b \pi y^2\, dx = \int_a^b \pi [f(x)]^2\, dx.$$

In other words, if the cross-section is a disk, we find the radius of the disk (in terms of x or y) and use $A = \pi(\text{radius})^2$.

The Washer Method The washer method is used to find volume when a region between two curves is revolved about an external axis perpendicular to the approximating strip. In particular, suppose f and g are continuous functions on $[a, b]$ with $f(x) \geq g(x) \geq 0$. Then if R is the region bounded above by $y = f(x)$, below by $y = g(x)$, and on the sides by $x = a$ and $x = b$, the solid formed by revolving R about the x-axis has volume

$$V = \int_a^b \pi [f(x)^2 - g(x)^2]\, dx.$$

In other words, if the cross-section is a washer, we find the inner radius r_{in} and outer radius r_{out} from a sketch and compute the area of the washer by subtracting the area of the inner disk from the area of the outer disk:

$$A = \pi(\text{outer radius})^2 - \pi(\text{inner radius})^2.$$

Method of Cylindrical Shells The shell method is used to find a volume generated when a region R is revolved about an axis L parallel to a typical approximating strip in R. For example, if R is the region bounded by the curve $y = f(x)$, the x-axis, and the vertical lines $x = a$ and $x = b$ where $0 \leq a \leq b$, then the solid generated by revolving R about the y-axis has volume

$$V = \int_a^b 2\pi x f(x) dx.$$