Math 201

Review for Test 3 Key

1. Let $f(x)=\sqrt{x}$ and $x_0=25$.

Then $f'(x)=\dfrac{1}{2\sqrt{x}}$ and so

$$f(x) \approx f(x_0)+f'(x_0)(x-x_0)$$
$$=\sqrt{25}+\frac{1}{\sqrt{25}}(x-25)$$
$$=5+\frac{1}{5}(x-25)$$

$$\sqrt{24}=f(24)\approx 5+\frac{1}{5}(24-25)$$
$$=5-\frac{1}{5}=4.8$$

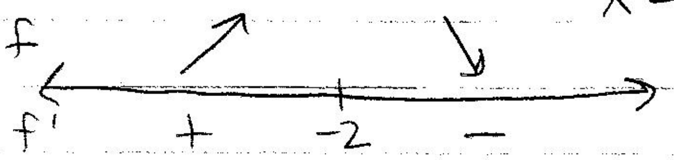
2. $f(x)=5-4x-x^2$

$f'(x)=-4-2x$

$f'(x)=0$ when $-2x-4=0$

$-2x=4$

$x=-2$



$f''(x)=-2$

no inflection points

$$f(x) = \frac{x}{x^2+2}$$

$$f'(x) = \frac{(x^2+2)(1)-(x)(2x)}{(x^2+2)^2}$$

$$= \frac{x^2+2-2x^2}{(x^2+2)^2} = \frac{-x^2+2}{(x^2+2)^2}$$

$f'(x)=0$ when
$-x^2+2=0$
$x=\pm\sqrt{2}$

$f'(x)$ dne when
$(x^2+2)^2=0$
no critical points here

f: $\searrow$ $\quad$ $\nearrow$ $\quad$ $\searrow$

f': $-$ $\quad -\sqrt{2} \quad$ $+$ $\quad \sqrt{2} \quad$ $-$

$$f''(x) = \frac{(x^2+2)^2(-2x)-(-x^2+2)(2(x^2+2)(2x))}{(x^2+2)^4}$$

$$= \frac{(x^2+2)(-2x)-(-x^2+2)(4x)}{(x^2+2)^3}$$

$$= \frac{(-2x^3-4x)-(-4x^3+8x)}{(x^2+2)^3}$$

$$= \frac{2x^3-12x}{(x^2+2)^3}$$

$f''(x)=0$ when
$2x^3-12x=0$
$2x(x^2-6)=0$
$x=0,\ x=\pm\sqrt{6}$

$f''(x)$ dne when
$(x^2+2)^3=0$
imaginary roots

f: $\frown$ $\quad$ $\smile$ $\quad$ $\frown$ $\quad$ $\smile$

f'': $-$ $\quad -\sqrt{6} \quad$ $+$ $\quad 0 \quad$ $-$ $\quad \sqrt{6} \quad$ $+$

inflection points at
$x=0, -\sqrt{6}, \sqrt{6}$

$f(x) = x^{2/3} - x$

$f'(x) = \frac{2}{3}x^{-1/3} - 1$

$= \dfrac{2}{3x^{1/3}} - 1 = \dfrac{2 - 3x^{1/3}}{3x^{1/3}}$

$f'(x) = 0$ when
$2 - 3x^{1/3} = 0$
$x^{1/3} = 2/3$
$x = 8/27$

$f'(x)$ dne when
$3x^{1/3} = 0$
$x = 0$

$\searrow \qquad \nearrow \qquad \searrow$

$- \quad 0 \quad + \quad 8/27 \quad -$

$f''(x) = -2/9\, x^{-4/3} = \dfrac{-2}{9x^{4/3}}$

$f''(x) = 0$ when
$-2 = 0$
$\uparrow$
never happens

$f''(x)$ dne when
$9x^{4/3} = 0$
$x = 0$

$\frown \qquad \frown$

$- \quad 0 \quad -$

no inflection points

3. $f(x) = \dfrac{x^2}{x^3 + 8}$

$f'(x) = \dfrac{(x^3+8)(2x) - (x^2)(3x^2)}{(x^3+8)^2}$

$= \dfrac{(2x^4 + 16x) - (3x^4)}{(x^3+8)^2}$

$= -x^4 + 16x \,/\, (x^3+8)^2$

$f'(x)=0$ when
$-x^4+16x=0$
$-x(x^3-16)=0$
$x=0$, $x=\sqrt[3]{16}$

$f'(x)$ dne when
$(x^3+8)^2=0$
$x=-\sqrt[3]{8}=-2$
↑
not in domain of f

critical points: $x=0$, $x=\sqrt[3]{16}$

4. $f'(x)=4x^3-9x$
$f'(x)=0$ when $4x^3-9x=0$
$x(4x^2-9)=0$
$x=0$, $x=\pm 3/2$

f	↘		↗		↘		↗
f'	$-$	$-3/2$	$+$	0	$-$	$3/2$	$+$

relative max at $x=0$
relative min at $x=-3/2$ and $x=3/2$

5. (a) $f(x)=x(x-4)^3$
$f'(x)=x(3(x-4)^2)+(x-4)^3$
$=(x-4)^2(3x+x-4)$
$=(x-4)^2(4x-4)$
$f'(x)=0$ when $(x-4)^2(4x-4)=0$
$x=4$ or $x=1$

↘		↗		↗
$-$	1	$+$	4	$+$

relative min at $x=1$

(b) $f(x) = 2x + 3x^{1/3}$

$f'(x) = 2 + x^{-2/3} = 2 + \dfrac{1}{x^{2/3}} = \dfrac{2x^{2/3}+1}{x^{2/3}}$

$f'(x) = 0$ when
$2x^{2/3} + 1 = 0$
$x^{2/3} = -1/2$
never happens

$f'(x)$ dne when
$x^{2/3} = 0$
$x = 0$

$\nearrow \quad \nearrow$

$+ \quad 0 \quad +$

no relative extrema

6. $f(x) = 2 - x + 2x^2 - x^3$

$f'(x) = -1 + 4x - 3x^2$

$f'(x) = 0$ when $-3x^2 + 4x - 1 = 0$

$(-3x^2 + 3x) + (x - 1) = 0$

$-3x(x-1) + (x-1) = 0$

$(x-1)(-3x+1) = 0$

$x = 1, \quad x = 1/3$

$\searrow \quad \nearrow \quad \searrow$

$- \quad 1/3 \quad + \quad 1 \quad -$

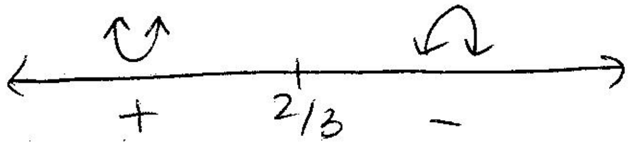
relative min at $x = 1/3$: $(1/3, 50/27)$

relative max at $x = 1$: $(1, 2)$

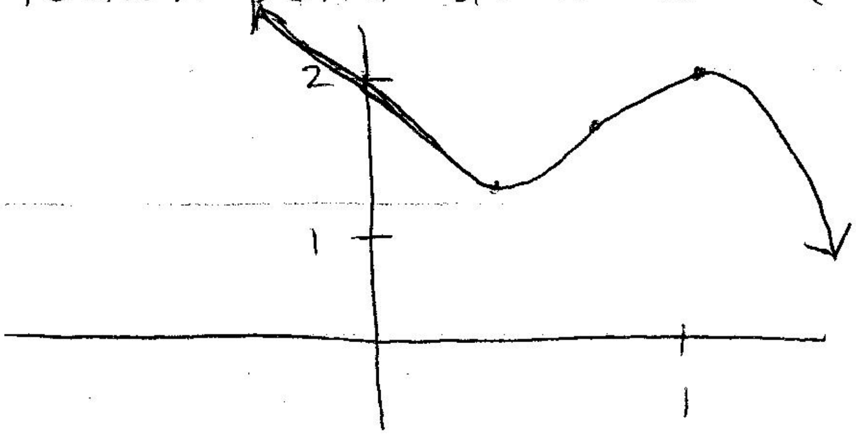
$f''(x) = 4 - 6x$

$f''(x) = 0$ when $4 - 6x = 0$

$x = 2/3$



inflection point at $x = 2/3$: $(2/3, 52/27)$



7. (a) $f(x) = 8x - x^2 \quad [0, 6]$

$f'(x) = 8 - 2x$

$f'(x) = 0$ when $8 - 2x = 0$

$x = 4$

$f(0) = 0$ - absolute min

$f(6) = 12$

$f(4) = 16$ - absolute max.

(b) $f(x) = (x^2 + x)^{2/3} \quad [-2, 3]$

$f'(x) = 2/3 (x^2 + x)^{-1/3} (2x + 1)$

$$= \frac{4x + 2}{3(x^2 + x)^{1/3}}$$

$f'(x) = 0$ when

$4x + 2 = 0$

$x = -1/2$

$f'(x)$ dne when

$3(x^2 + x)^{1/3} = 0$

$x^2 + x = 0$

$x(x + 1) = 0$

$x = 0, \; x = -1$

$f(-2) = \sqrt[3]{4}$

$f(3) = \sqrt[3]{144}$ – absolute max

$f(-1/2) = \sqrt[3]{1/16}$

$f(0) = 0$ – absolute min.

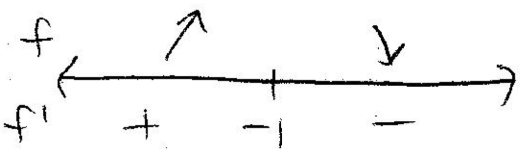
$f(-1) = 1$

8. (a) $f(x) = 3 - 4x - 2x^2$ $\quad (-\infty, \infty)$

$f'(x) = -4 - 4x$

$f'(x) = 0$ when $-4 - 4x = 0$

$x = -1$



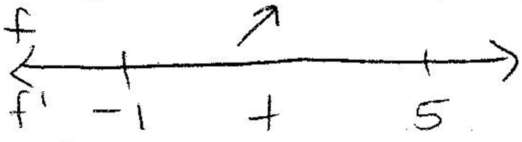
$x = -1$ is an absolute max; there is no absolute min.

(b) $f(x) = \dfrac{x-2}{x+1}$ $\quad (-1, 5]$

$$f'(x) = \frac{(x+1)(1) - (x-2)(1)}{(x+1)^2}$$

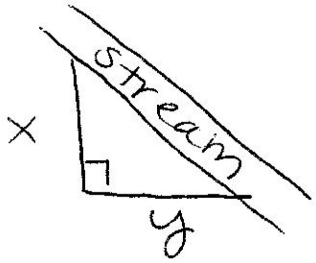
$$= \frac{(x+1) - (x-2)}{(x+1)^2}$$

$$= \frac{3}{(x+1)^2}$$ ← no critical points in $(-1, 5]$



Since f is increasing on $(-1, 5]$, f will have an absolute max at $x = 5$, but no absolute min.

9.



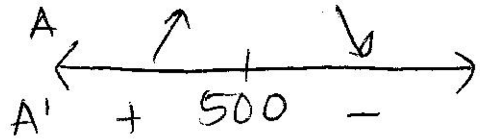
$1000 = x + y \rightarrow y = 1000 - x$
want to maximize area

$A = \frac{1}{2} y x = \frac{1}{2}(1000 - x)(x) = 500x - \frac{1}{2}x^2$

$A' = 500 - x$

$A' = 0$ when $500 - x = 0$

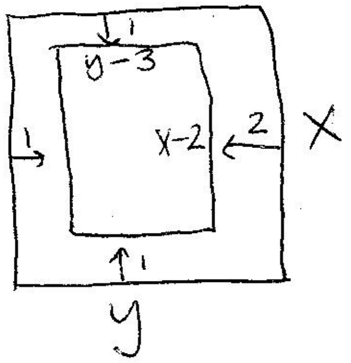
$x = 500$



$x = 500$ is an absolute maximum

$\hookrightarrow y = 1000 - x = 1000 - 500 = 500$ ft.

10.



Printable area is 42 in^2

$\hookrightarrow (x-2)(y-3) = 42$

Least paper used
↳ minimize perimeter

$(x-2)(y-3) = 42$

$y - 3 = \frac{42}{x-2}$

$y = \frac{42}{x-2} + 3$

$P = 2x + 2y$

$= 2x + 2\left(\frac{42}{x-2} + 3\right)$

$$P = 2x + \frac{84}{x-2} + 6$$

$$P' = 2 - \frac{84}{(x-2)^2} = \frac{2(x-2)^2 - 84}{(x-2)^2}$$

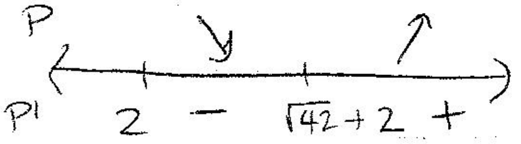
$P' = 0$ when

$$2(x-2)^2 - 84 = 0$$
$$(x-2)^2 = 42$$
$$x - 2 = \sqrt{42}$$
$$x = \sqrt{42} + 2$$

P' dne when

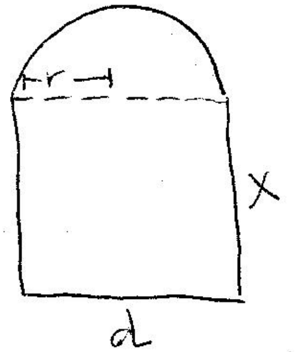
$$(x-2)^2 = 0$$
$$x = 2$$

$\uparrow$ doesn't make sense in application



$$x = \sqrt{42} + 2 \approx 8.48074 \text{ in.}$$
$$y = \frac{42}{x-2} + 3 = \sqrt{42} + 3 \approx 9.48074 \text{ in.}$$

11.



$d = 2r$

$$P = \underbrace{\tfrac{1}{2}(\pi d)}_{\text{Perimeter of semi-circle}} + \underbrace{d + x + x}_{\text{perimeter of rectangle}}$$
$$= \tfrac{1}{2}\pi d + d + 2x$$

want to maximize area.

$$A = \underbrace{\tfrac{1}{2}(\pi r^2)}_{\text{area of semi-circle}} + \underbrace{xd}_{\text{area of rectangle}}$$
$$= \tfrac{1}{2}(\pi r^2) + \left(\tfrac{1}{2}\left(P - \tfrac{1}{2}\pi d - d\right) d\right)$$

$$= \tfrac{1}{2}\pi r^2 + \tfrac{1}{2}\left(P - \tfrac{1}{2}\pi(2r) - 2r\right)(2r)$$
$$= \tfrac{1}{2}\pi r^2 + Pr - \pi r^2 - 2r^2$$

$A' = \pi r + P - 2\pi r - 4r$

$A' = 0$ when $\pi r + P - 2\pi r - 4r = 0$

$$P = 2\pi r + 4r - \pi r$$
$$P = r(2\pi + 4 - \pi)$$
$$r = \frac{P}{\pi + 4}$$

$A'' = -\pi - 4 < 0$

$\hookrightarrow$ By 2nd derivative test, $r = \dfrac{P}{\pi + 4}$ is a max.

12. $f(x) = x^3 + x - 1$ $\qquad x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

$f'(x) = 3x^2 + 1$

initial guess: $x_1 = 1$

$$x_2 = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{1}{4} = .75$$

$$x_3 = .75 - \frac{f(.75)}{f'(.75)} = .75 - \frac{.1719}{2.6875} = .6860$$

$$x_4 = .6860 - \frac{f(.6860)}{f'(.6860)} = .6860 - \frac{.0089}{2.4118} = .6823$$

$$x_5 = .6823 - \frac{f(.6823)}{f'(.6823)} = .6823 + \frac{.0001}{2.3966} = .6823$$

Approximate answer: $x = .6823$

13. $f(x) = x\sin(x)$ $[0, \pi]$

$f'(x) = x\cos(x) + \sin(x)$ want $f'(x) = 0$

$f''(x) = -x\sin(x) + 2\cos(x)$

Initial guess: $x_1 = 2$ (look at graph)

$x_2 = 2 - \frac{f'(2)}{f''(2)} = 2 - \frac{.0770}{-2.6509} = 2.0291$

$x_3 = 2.0291 - \frac{f'(2.0291)}{f''(2.0291)} = 2.0291 - \frac{-.0008}{-2.7046} =$

2.0288

$x_4 = 2.0288 - \frac{f'(2.0288)}{f''(2.0288)} = 2.0288 - \frac{-.0001}{-2.7040} =$

2.0288

Max occurs at $x = 2.0288$

Absolute max is $f(2.0288) = 1.81971$

14. $f(x) = x^3 - 3x^2 + 2x$ $[0, 2]$

f is continuous and differentiable

$f(0) = 0$, $f(2) = 8 - 12 + 4 = 0$

So there exists c in $(0,2)$ such that $f'(x) = 0$:

$f'(x) = 3x^2 - 6x + 2$

$f'(x) = 0$ when $3x^2 - 6x + 2 = 0$

$$x = \frac{6 \pm \sqrt{36 - 24}}{6} = \frac{6 \pm \sqrt{12}}{6}$$

$$= \frac{6 \pm 2\sqrt{3}}{6} \approx .4227, 1.5774$$

$$c = \frac{6 + 2\sqrt{3}}{6} + \frac{6 - 2\sqrt{3}}{6}$$

15. $f(x) = x^3 + x - 4 \qquad [-1,2]$

f is continuous and differentiable, so there exists c in (-1,2) such that

$$f'(c) = \frac{f(2)-f(-1)}{2-(-1)} = \frac{6+6}{3} = 4$$

$f'(x) = 3x^2 + 1$

$f'(x) = 4$ when $3x^2 + 1 = 4$

$3x^2 = 3$

$x^2 = 1$

$x = 1, -1 \rightarrow \boxed{c = 1}$

16. True since

$$\text{avg. rate of change} = \frac{f(b)-f(a)}{b-a}$$

$$\text{instan. rate of change at } x = f'(x)$$

So by Mean-Value Theorem, there is a c in (a,b) such that

$$f'(c) = \frac{f(b)-f(a)}{b-a};$$

ie, instan. rate of change at c = average rate of change

17. Let $f(x) = 3x^4 + x^2 - 4x$. Then f is continuous and differentiable, and $f(0) = 0$ and $f(1) = 0$. So, by Rolle's Theorem, there is a c in $(0,1)$ such that $f'(c) = 0$. In othere words, $f'(x) = 12x^3 + 2x - 4 = 0$ has a solution in $(0,1)$.