Math 201
Review for Test 2 Key

1. (a)

$$\text{avg. rate of change} = \frac{f(x_1)-f(x_0)}{x_1-x_0} = \frac{(2)^3-(1)^3}{2-1} = 7$$

(b) instan. rate of change (specific x_0)

$$= \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}$$

$$= \lim_{h\to 0} \frac{f(1+h)-f(1)}{h}$$

$$= \lim_{h\to 0} \frac{(1+h)^3-(1)^3}{h}$$

$$= \lim_{h\to 0} \frac{h^3+3h^2+3h+1-1}{h}$$

$$= \lim_{h\to 0} \frac{h(h^2+3h+3)}{h}$$

$$= \lim_{h\to 0} h^2+3h+3 = 3$$

(c) instan. rate of change (arbitrary x_0)

$$= \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}$$

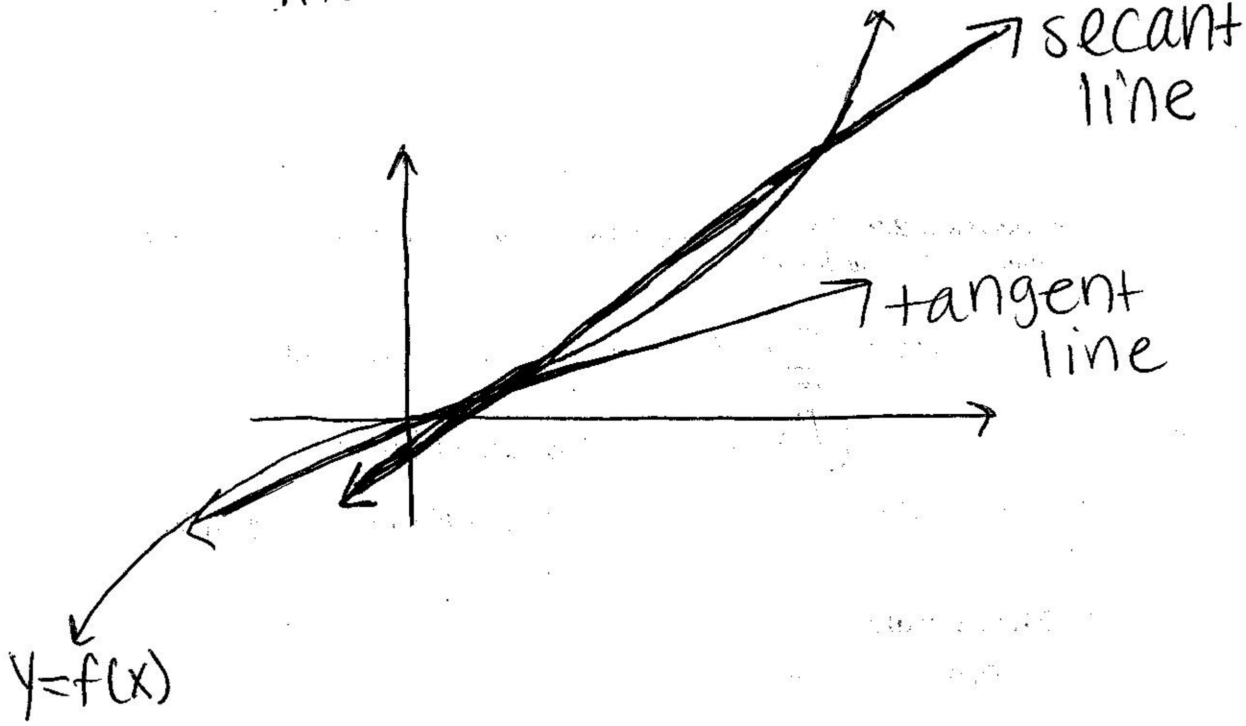
$$= \lim_{h\to 0} \frac{(x_0+h)^3-x_0^3}{h}$$

$$= \lim_{h\to 0} \frac{x_0^3+3x_0^2h+3x_0h^2+h^3-x_0^3}{h}$$

$$= \lim_{h\to 0} \frac{h(3x_0^2+3x_0h+h^2)}{h}$$

$$= \lim_{h\to 0} 3x_0^2 + 3x_0 h + h^2 = 3x_0^2$$

(d)



2. (a)

slope of tangent line $= f'(x_0) = 2x_0 + 3$

(b) slope of tangent line at $x_0 = 2$ $= f'(2) = 2(2)+3 = 7$

3.

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{(x+h)+1} - \frac{1}{x+1}}{h}$$

$$= \lim_{h\to 0} \frac{\frac{(x+1)-(x+h+1)}{(x+1)(x+h+1)}}{h}$$

$$= \lim_{h\to 0} \frac{-h}{h(x+1)(x+h+1)} = \lim_{h\to 0} \frac{-1}{(x+1)(x+h+1)}$$

$$= \frac{-1}{(x+1)^2}$$

4. (a) True by Theorem 2.2.3 on page 127.

(b) If $f(x)$ is a cubic polynomial, then
$f(x) = ax^3 + bx^2 + cx + d$ which implies
$f'(x) = 3ax^2 + 2bx + c$. Therefore, the statement is true.

(c) $g(x) = f(x)\sin(x)$ gives
$g'(x) = f'(x)\sin(x) + f(x)\cos(x)$ so
$g'(0) = f'(0)\sin(0) + f(0)\cos(0) = f(0)$.

The statement is true.

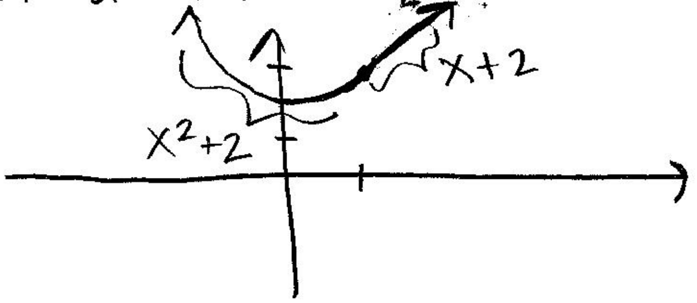
5. $f(x)$ is continuous since:

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x^2+2 = 3$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} x+2 = 3$$

so $\lim_{x\to 1} f(x) = 3 = f(1)$.

However, approaching $x=1$ from the left gives a tangent line with slope $2(1)=2$ and approaching $x=1$ from the right gives a tangent line with slope 1. Thus, f is not differentiable at $x=1$.



6. (a) $f(x) = x^{1/2} + x^{-1}$

$f'(x) = 1/2\, x^{-1/2} - x^{-2}$

(b) $f'(x) = (2 - x - 3x^3)(5x^4) + (-1 - 9x^2)(7 + x^5)$

(c) $f'(x) = \dfrac{(x^4 + x + 1)(1) - (x-2)(4x^3 + 1)}{(x^4 + x + 1)^2}$

(d) $f(x) = (2x^{1/2} + 1)\left(\dfrac{2-x}{x^2+3x}\right)$

$f'(x) = (2x^{1/2} + 1)\left(\dfrac{(x^2+3x)(-1) - (2-x)(2x+3)}{(x^2+3x)^2}\right) +$

$\left(\dfrac{2-x}{x^2+3x}\right)(x^{-1/2})$

(e) $f'(x) = \dfrac{(x^2 + \sin(x))(\cos(x)) - \sin(x)(2x + \cos(x))}{(x^2 + \sin(x))^2}$

(f) $f(x) = (x^5 - x + 1)^{-9}$

$f'(x) = -9(x^5 - x + 1)^{-10}(5x^4 - 1)$

(g) $f'(x) = 4 + 20\sin^3(x)\cos(x)$

7. (a) $y' = 28x^6 - 15x^2 + 2$

$y'' = 168x^5 - 30x$

(b) $y' = -x^2\sin(x) + 2x\cos(x) + 4\cos(x)$

$y'' = -x^2\cos(x) - 2x\sin(x) - 2x\sin(x) + 2\cos(x) - 4\sin(x)$

(c) $y' = 6x\cos(3x^2)$

$y'' = -36x^2\sin(3x^2) + 6\cos(3x^2)$

8. G, m, M are constants, so

$$\frac{dF}{dr} = -2GmMr^{-3} = \frac{-2GmM}{r^3}.$$

9. (a) $F'(x) = 6f'(x) - 5g'(x)$

$$F'(\pi) = 6f'(\pi) - 5g'(\pi)$$
$$= 6(-1) - 5(2) = -6 - 10 = -16$$

(b) $F'(x) = x(f'(x) + g'(x)) + (1)(f(x) + g(x))$

$$F'(\pi) = \pi(f'(\pi) + g'(\pi)) + f(\pi) + g(\pi)$$
$$= \pi(-1 + 2) + 10 - 3$$
$$= \pi + 7$$

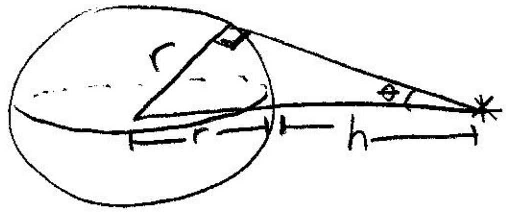
(c) $F'(x) = 2f(x)g'(x) + 2f'(x)g(x)$

$$F'(\pi) = 2f(\pi)g'(\pi) + 2f'(\pi)g(\pi)$$
$$= 2(10)(2) + 2(-1)(-3)$$
$$= 40 + 6 = 46$$

(d) $F'(x) = \dfrac{(4+g(x))f'(x) - f(x)g'(x)}{(4+g(x))^2}$

$$F'(\pi) = \frac{(4+g(\pi))f'(\pi) - f(\pi)g'(\pi)}{(4+g(\pi))^2}$$
$$= \frac{(4-3)(-1) - (10)(2)}{(4-3)^2}$$
$$= \frac{-1-20}{1} = -21$$

10.



(a)

$$\sin(\theta) = \frac{r}{r+h}$$

$$(r+h)\sin(\theta) = r$$

$$h(\sin(\theta)) = r - r\sin(\theta)$$

$$h = \frac{r(1-\sin(\theta))}{\sin(\theta)} = r(\csc(\theta)-1)$$

(b) $$h = \frac{6378(1-\sin(\theta))}{\sin(\theta)}$$

$$\frac{dh}{d\theta} = \frac{\sin(\theta)(-6378\cos(\theta)) - 6378(1-\sin(\theta))\cos(\theta)}{\sin^2(\theta)}$$

$$\left.\frac{dh}{d\theta}\right|_{\theta=30°} = \frac{1/2(-6378(\sqrt{3}/2)) - 6378(1-1/2)\sqrt{3}/2}{(1/2)^2}$$

$$= \frac{\frac{-6378\sqrt{3}}{4} - \frac{6378\sqrt{3}}{4}}{1/4}$$

$$= -12756\sqrt{3} \text{ km/rad} \cdot \frac{\pi \text{ rad}}{180 \text{ deg}}$$

$$= \frac{-12756\pi\sqrt{3}}{180} \text{ km/deg.}$$

11. $y = \cos(x)$

$y' = -\sin(x)$

$y'' = -\cos(x)$

$y''' = \sin(x)$

$y^{iv} = \cos(x)$

$100 \div 4 = 25$ (no remainder)

so

$$\frac{d^{100}}{dx^{100}}[\cos(x)] = \cos(x)$$

12. $F'(x) = f'(g(x)) \cdot g'(x)$ (by Chain Rule)

$= \sqrt{3(x^2-1)+4}\,(2x)$

13. (a) $x^3(2y)\frac{dy}{dx} + 3x^2y^2 - 5x^2\frac{dy}{dx} - 10xy + 1 = 0$

$$\frac{dy}{dx}(2x^3y - 5x^2) = 10xy - 3x^2y^2 - 1$$

$$\frac{dy}{dx} = \frac{10xy - 3x^2y^2 - 1}{2x^3y - 5x^2}$$

(b) $-\sin(xy^2)\left(2xy\frac{dy}{dx} + y^2\right) = \frac{dy}{dx}$

$$-y^2\sin(xy^2) = \frac{dy}{dx}\left(1 + 2xy\sin(xy^2)\right)$$

$$\frac{dy}{dx} = \frac{-y^2\sin(xy^2)}{1+2xy\sin(xy^2)}$$

14. $3y^2\frac{dy}{dx} + 2yx + x^2\frac{dy}{dx} + 2x - 6y\frac{dy}{dx} = 0$

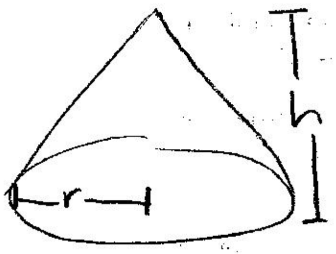
Plugging in (0,3):

$$27\frac{dy}{dx} - 18\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = 0$$

slope of tangent line at $(0,3)$ $= 0$.

15.



Given

$\frac{dV}{dt} = 8$

$h = 6$

Find

$\frac{dh}{dt} = ?$

Formula

$h = 2r$

$V = \frac{1}{3}\pi r^2 h$

since we're not given any information about the radius, except that $r = \frac{1}{2}h$, we substitute this into our volume formula to get volume in terms of height:

$$V = \frac{1}{3}\pi\left(\frac{1}{2}h\right)^2 h$$

$$V = \frac{1}{12}\pi h^3$$

Differentiating implicitly, we get

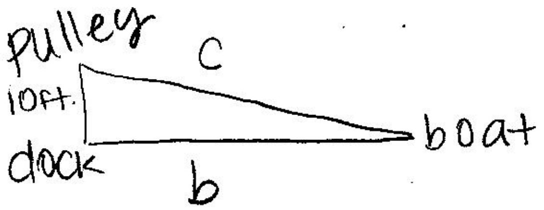
$$\frac{dV}{dt} = \frac{1}{4}\pi h^2 \frac{dh}{dt}$$

substituting our knowns, we have

$$8 = \frac{1}{4}\pi (6)^2 \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{32}{36\pi} \text{ ft/min} \approx .2829 \text{ ft/min.}$$

16.



Given

$\frac{dc}{dt} = 20$

$c = 125$

$b = \sqrt{(125)^2 - 100}$

Find

$\frac{db}{dt} = ?$

Formula

$(10)^2 + b^2 = c^2$

Differentiating implicitly:

$2c\frac{dc}{dt} = 2b\frac{db}{dt}$

Plugging in knowns:
(Note: To find b, we used $(10)^2 + b^2 = c^2$ and $c = 125$.)

$125(20) = \sqrt{15525}\frac{db}{dt}$

$\frac{db}{dt} = \frac{2500}{\sqrt{15525}}$ ft/min ≈ 20.0643 ft/min.