Finding the Zero of a Polynomial

Consider the polynomial $P(x) = x^3 + 3x - 1$. Your goal is to find a zero of this function: ie, a number a so that $P(a) = 0$. Although there is an algebraic technique for finding a zero of a cubic polynomial, we are going to approximate a zero. We want the approximation to be within 10^{-2} of an actual zero.

To begin, show that the equation $P(x) = 0$ has at least one solution in the interval $[-1, 1]$. You must give a good justification that such a solution exists.

One way to approximate a solution is to bisect the interval $[-1, 1]$, determine whether $P(x) = 0$ has a solution in $[-1, 0]$ or $[0, 1]$ (as you did in the previous step), and then repeat the process with the new interval containing a solution. How many times must you repeat the bisection process to have a sufficiently accurate answer? Find such an answer. Be sure you understand what it means to bisect an interval.

Calculus gives us another way to perform the search. We use the idea that the tangent line is a good approximation to the graph of a function.

Let $y = f(x)$ be a function of x. What is the equation of the line tangent to the graph of f at $(a, f(a))$? What is the x-intercept of this tangent line? Draw a picture to demonstrate what is going on.

Now return to the original problem. Begin with one of the end points of the original interval; this is your original guess. Apply the operation you demonstrated in the previous step, obtaining the x-intercept of the tangent line as a new guess, which hopefully is a better approximation to a solution of the equation $x^3 + 3x - 1 = 0$ than the end point you started with. Is this answer within the desired margin of error from the answer you obtained using the method of bisection? When you get an answer within the margin of error, you may stop. Otherwise, repeat the operation, this time beginning with your latest guess.

Compare the two techniques for finding a solution. Which is easier to understand? Why? Which is faster - that is, which leads to an answer within the desired degree of accuracy in the fewest number of iterations?