**Math 201**

**Section 3.6** L'Hopital's Rule; Indeterminate Forms

Indeterminate form of type $\frac{0}{0}$

If we have a limit of the form

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

where both $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$, then this limit may or may not exist and is called an indeterminate form of type $\frac{0}{0}$.

Indeterminate form of type $\frac{\infty}{\infty}$

If we have a limit of the form

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

where both $f(x) \to \infty$ (or $-\infty$) and $g(x) \to \infty$ (or $-\infty$), then the limit may or may not exist and is called an indeterminate form of type $\frac{\infty}{\infty}$.

L'Hopital's Rule

Suppose $f$ and $g$ are differentiable and $g'(x) \neq 0$ near $a$ (except possibly at $a$). Suppose that

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

is an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right side exists (or is $\infty$ or $-\infty$).

**Note**: L'Hopital's Rule is also valid for one-sided limits and for limits at infinity or negative infinity (ie, "$x \to a$ " can be replaced by $x \to a^+$, $x \to a^-$, $x \to \infty$, or $x \to -\infty$).

Indeterminate Products

If $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = \infty$ (or $-\infty$), then it isn't clear what the value of $\lim_{x \to a} f(x)g(x)$, if any, will be. This kind of limit is called an indeterminate form of type $0 \cdot \infty$. We can deal with it by writing the product $fg$ as a quotient:

$$fg = \frac{f}{1/g} \text{ or } \frac{g}{1/f}.$$

This converts the given limit into an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$ so that we can use l'Hopital's Rule.

Indeterminate Differences

If $\lim_{x \to a} f(x) = \infty$ and $\lim_{x \to a} g(x) = \infty$, then the limit $\lim_{x \to a}[f(x) - g(x)]$ is called an indeterminate form of type $\infty - \infty$. We try to convert the difference into a quotient (for ex: by using a common denominator, rationalization, factoring out a common factor, etc.) so that we have an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Indeterminate Powers

Several indeterminate forms arise from the limit

$$\lim_{x\to a}[f(x)]^{g(x)}$$

1. $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$ (type $0^0$)
2. $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = 0$ (type $\infty^0$)
3. $\lim_{x\to a} f(x) = 1$ and $\lim_{x\to a} g(x) = \pm\infty$ (type $1^\infty$)

Each of these 3 cases can be treated by taking the natural logarithm:

Let $y = [f(x)]^{g(x)}$, then $\ln(y) = g(x) \cdot \ln(f(x))$.