Differential Equations Seminar: Week 6 Solutions

1. Converting to polar, the system becomes

$$\begin{aligned} r' &= r(r^2-4) \\ \theta' &= 1. \end{aligned}$$

Then $r = 2$ is a periodic solution.

2. The only equilibrium point of the system is (0,0). Converting to polar, the system becomes

$$\begin{aligned} r' &= r^3(2\cos^2\theta + 1) - r \\ \theta' &= 1. \end{aligned}$$

Since $1 \le 2\cos^2\theta + 1 \le 3$, we have that

$$r'\big|_{r=2} = 8(2\cos^2\theta + 1) - 2 \ge 8 - 2 = 6 > 0$$

and

$$r'\big|_{r=\frac{1}{2}} = \frac{1}{8}(2\cos^2\theta + 1) - \frac{1}{2} \le \frac{1}{8}(3) - \frac{1}{2} = -\frac{1}{8} < 0.$$

The region $D = \{(x,y) : \frac{1}{2} \le r \le 2\}$ is negatively invariant (trajectories are trapped as $t \to -\infty$) and contains no equilibrium points, therefore it must contain at least one limit cycle.

3. The only equilibrium point of the system is (0,0). Converting to polar, the system becomes

$$\begin{aligned} r' &= r - r^3(1 + \tfrac{1}{2}\sin^2\theta) \\ \theta' &= 1. \end{aligned}$$

Since $1 \le 1 + \frac{1}{2}\sin^2\theta \le \frac{3}{2}$, we have that

$$r'\big|_{r=2} = 2 - 8(1 + \frac{1}{2}\sin^2\theta) \le 2 - 8 = -6 < 0$$

and

$$r'\big|_{r=\frac{1}{2}} = \frac{1}{2} - \frac{1}{8}(1 + \frac{1}{2}\sin^2\theta) \ge \frac{1}{2} - \frac{1}{8}(\frac{3}{2}) = \frac{5}{16} > 0.$$

The region $D = \{(x,y) : \frac{1}{2} \le r \le 2\}$ is positively invariant (trajectories are trapped as $t \to \infty$) and contains no equilibrium points, therefore it must contain at least one limit cycle.

4. The only equilibrium point of the system is (0,0). Converting to polar, the system becomes

$$\begin{aligned} r' &= r(3 - e^{r^2}) \\ \theta' &= 1. \end{aligned}$$

We have that

$$r'\big|_{r=1} = 3 - e > 0$$

and

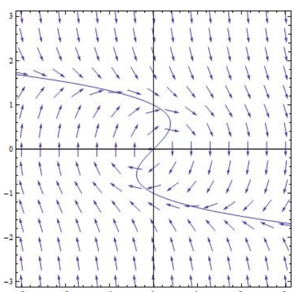
$$r'\big|_{r=2} = 2(3 - e^4) < 0.$$

The region $D = \{(x, y) : 1 \leq r \leq 2\}$ is positively invariant (trajectories are trapped as $t \to \infty$) and contains no equilibrium points, therefore it must contain at least one limit cycle.

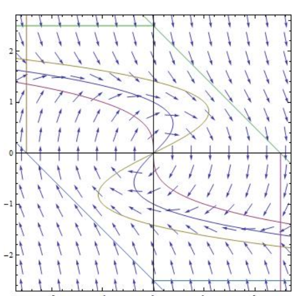
5. Upon converting the second-order equation into a system, we obtain

$$\begin{aligned} x' &= y \\ y' &= -x + y(1 - y^2). \end{aligned}$$

The only equilibrium point of the system is (0,0). The x-nullcline is $y = 0$, and the y-nullcline is $x = y - y^3$. Looking at a phase portrait together with the nullclines, we have:



Note that a rectangle will not serve as a trapping region for all trajectories, since trajectories near the top-right and bottom-left corners of any rectangle centered at the origin will be leaving the rectangle as $t \to \infty$. However, the following hexagonal region will be positively invariant:



Here, we have included two additional cubic functions, $x = -y^3$ and $x = 2y - y^3$, along which the slopes of all trajectories are ± 1 (recall that the slope of a trajectory $\frac{dy}{dx}$ can be computed by dividing $\frac{dy}{dt}$ by $\frac{dx}{dt}$). Any trajectory within the "band" of cubics has a slope less than 1 in magnitude. To see why this region is now positively invariant for all trajectories, consider a trajectory near the slanted upper-right edge of the region. This edge has slope -1, but since all trajectories near this edge are outside of the "band" of cubics, they will also have negative slopes larger in magnitude than -1, and hence will be entering the hexagonal region. A similar argument works for the bottom-left edge.

Furthermore, linearizing the system around the origin leads to a Jacobian matrix with eigenvalues $\lambda = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$, hence the origin is a spiral source. We can thus draw a sufficiently small circle around the origin on which all trajectories will be moving away from the origin.

The hexagonal region, excluding this neighborhood around the origin, will then be positively invariant and contain no critical points. Thus by the Poincaré-Bendixson Theorem, the nonlinear system has at least one limit cycle (implying that the original second-order equation has at least one periodic solution).