Differential Equations Seminar: Week 10 Solutions

1.

a) Uniform.

b) Local.

c) Uniform.

d) Local.

e) Uniform.

2. Suppose f is Lipschitz on D. Then for any $x_0 \in D$,

$$|f'(x_0)| = \lim_{x \to x_0} \left| \frac{f(x) - f(x_0)}{x - x_0} \right| = \lim_{x \to x_0} \frac{|f(x) - f(x_0)|}{|x - x_0|} \leq \lim_{x \to x_0} \frac{K|x - x_0|}{|x - x_0|} = K.$$

Hence $f'(x)$ is bounded by K for all $x \in D$.

Conversely, suppose $f'(x)$ is bounded by K for all $x \in D$. Since f is differentiable, we can apply the Mean Value Theorem; namely, for any pair of points $x, y \in D$ with $x < y$, we have

$$f(x) - f(y) = f'(c)(x - y).$$

for some $c \in (x, y)$. But then

$$|f(x) - f(y)| = |f'(c)||x - y| \leq K|x - y|.$$

Hence f is Lipschitz with constant K.